UNSTEADY RADIATIVE HEATING OF A CYLINDRICAL BODY

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We consider the heating of a cylinder ($0 \le r \le R$, $0 \le z \le L$), the upper end of which is subjected to radiative heating, while the remaining part of the surface exchanges heat with the surrounding medium in accordance with Newton's law.

The problem can be formulated mathematically as follows:

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial T}{\partial r}\right)+\frac{\partial^2 T}{\partial z^2}=\frac{1}{a}\frac{\partial T}{\partial \tau}\qquad \left(\begin{matrix}0\leqslant r\leqslant R\\ 0\leqslant z\leqslant L\end{matrix}\right),\quad \tau\geqslant 0\ ;$$

$$\left(\frac{\partial T}{\partial r}\right)_{r=R}=h\left[f(R, z, \tau)-T(R, z, \tau)\right],$$

$$\left(\frac{\partial T}{\partial z}\right)_{z=0}=h_1\left[T(r, 0, \tau)-f(r, 0, \tau)\right],$$

$$\left(\frac{\partial T}{\partial z}\right)_{z=L}=E(r, L, \tau),\quad T(r, z, 0)=T_0, \tag{1}$$

$$E(r, L, \tau)=H\left[T_1^4-T^4(r, L, \tau)\right].$$

Here a is the coefficient of thermal diffusivity, H is the relative emission coefficient, T_1 is the temperature of the radiator, T_0 is the initial temperature of the body, $f(r, z, \tau)$ is the temperature of the surrounding medium, and h and h_1 are the relative coefficients of heat transfer for the corresponding surfaces.

The Laplace transformation reduces the boundary-value problem (1) to the following boundary-value problem in the transformed variables:

$$\frac{1}{r}\left[\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right)-\frac{p}{a}ru\right]+\frac{\partial^2 u}{\partial z^2}=-\frac{T_0}{a},$$

$$\left(\frac{\partial u}{\partial r}+hu\right)_{r=R}=hF(r, z, p),$$

$$\left(\frac{\partial u}{\partial z}-h_1u\right)_{z=0}=h_1F(r, z, p), \tag{2}$$

$$\left(\frac{\partial u}{\partial z}\right)_{z=L}=E_1(r, L, p).$$

Here

$$F=\int_0^\infty f(r, z, \tau)e^{-p\tau}d\tau,\quad u=\int_0^\infty T(r, z, \tau)e^{-p\tau}d\tau,$$

$$E_1=\int_0^\infty E(r, L, \tau)e^{-p\tau}d\tau.$$

We multiply Eq. (2) by $J_0(\mu_n r/R)dr$ and integrate from 0 to R. The boundary-value problem (2) becomes

$$\frac{d^2v_1}{dz^2}-\varkappa v_1=\Phi(z, p),\quad \left(\frac{dv_1}{dz}-h_1v_1\right)_{z=0}=h_1F_1(0, p),$$

$$\left(\frac{dv_1}{dz}\right)_{z=L}=E_2(L, p). \tag{3}$$

Here

$$v_1(z)=\int_0^R ru(r, z, p)J_0\left(\frac{\mu_n r}{R}\right)dr,$$

$$F_1(z, p)=\int_0^R rF(r, z, p)J_0\left(\frac{\mu_n r}{R}\right)dr,$$

$$\mu_n=\left(\varkappa_n-\frac{pR}{a}\right)^{1/2},$$

$$E_2(L, p)=\int_0^R rE_1(r, L, p)J_0\left(\frac{\mu_n r}{R}\right)dr,$$

$$\Phi(z, p)=hRJ_0(\mu_n)F(R, z, p)+\frac{T_0R^2}{a\mu_n}J_1(\mu_n).$$

The functions $J_0(\mu_n r/R)$ are the eigenfunctions of the boundary-value problem

$$\frac{1}{r}\frac{d}{dr}\left(r\frac{dy}{dr}\right)+\left(\varkappa-\frac{p}{a}\right)y=0,\qquad \left(\frac{dy}{dr}+hy\right)_{r=R}=0. \tag{4}$$

The eigenvalues are defined by the equation

$$\mu_nJ_1(\mu_n)-RhJ_0(\mu_n)=0.$$

Without loss of generality we can set

$$f(r, z, \tau)=f(\tau)=T_0-k\tau.$$

The solution of (3) can be arranged in the form

$$v_1(z, p)=\frac{T_0R^2J_1(\mu_n)}{a\mu_n\gamma_n^2}+hRJ_0(\mu_n)\frac{F(p)}{\gamma_n^2}-$$

$$-\frac{h_1T_0R^2J_1(\mu_n)\operatorname{ch}\gamma_n(L-z)}{a\mu_n\gamma_n^2\omega(\gamma_n)}+$$

$$+\left[\frac{R^2J_1(\mu_n)}{\mu_n}-\frac{hRJ_0(\mu_n)}{\gamma_n^2}\right]\frac{h_1F(p)\operatorname{ch}\gamma_n(L-z)}{\omega(\gamma_n)}+$$

$$+\frac{1}{\gamma_n\omega(\gamma_n)}(h_1\operatorname{sh}\gamma_n z+\gamma_n\operatorname{ch}\gamma_n z)\int_0^R\xi J_0(\mu_n\xi)E_1(\xi, L, p)d\xi. \tag{5}$$

Here $\gamma_n=(\mu_n^2/R^2+p/a)^{1/2}$ and $\omega(\gamma_n)=\gamma_n\operatorname{sh}\gamma_n L+h_1\operatorname{ch}\gamma_n L$.

It was shown in [1] that $v_1(z, p)$ denotes the Fourier coefficients for the expansion of $v(r, z, p)$ in an orthonormalized system of eigenfunctions

$$\frac{1}{N_n}J_0(\mu_n r/R);\quad N_n=\frac{R^2}{2}\left[J_0^2(\mu_n)+J_1^2(\mu_n)\right]$$

is the norm of the eigenfunctions.

Then the solution of the boundary-value problem for the transformed variables is

$$v(r, z, p)=\sum_{n=1}^\infty v_1(z, p)J_0(\mu_n r/R)/N_n. \tag{6}$$

The final solution of the problem in the original variables is obtained by applying the inverse Laplace transformation

$$T(r, z, \tau)=\frac{1}{2\pi i}\int_{\sigma-i\infty}^{\sigma+i\infty}u(r, z, p)e^{p\tau}dp. \tag{7}$$

If we assume that we can interchange the order of integration and summation, taking (6) into account, we obtain the following for the definition of the original of $v_1(z, p)$:

$$\frac{T_0R^2J_1(\mu_n)}{a\mu_n\gamma_n^2}\div\frac{T_0R^2J_1(\mu_n)}{\mu_n}e^{-a\mu_n\tau/R^2};$$

$$\Psi_1(\tau)=\frac{h_1T_0R^2J_1(\mu_n)}{a\mu_n}\frac{1}{2\pi i}\int_{\sigma-i\infty}^{\sigma+i\infty}\frac{\operatorname{ch}\gamma_n(L-z)}{\gamma_n^2\omega(\gamma_n)}e^{p\tau}dp. \tag{8}$$

We put $s=a\gamma_n^2$ and obtain

$$\Psi_1(\tau)=\frac{T_0R^2J_1(\mu_n)}{\mu_n}e^{-a\mu_n^2\tau/R^2}\frac{1}{2\pi i}\int_{\sigma-i\infty}^{\sigma+i\infty}\frac{\operatorname{ch}\sqrt{s/a}(L-z)}{s\Omega(s)}e^{s\tau}ds,$$

$$\Omega(s)=\omega(s)/h_1. \tag{9}$$

The denominator of the integrand in (9) has an infinite set of poles s_m ($m = 0, 1, 2, \ldots$); $s_0 = 0$ and all the remaining roots are defined by the equation

$$\operatorname{ctg} \nu = \nu / (Lh_1), \quad \nu = i\,(s/a)^{1/2} L . \qquad (10)$$

By the theorem on expansion in the transformed variables [2], the original of the integrand in (9) is equal to the sum of the residues at all the poles

$$\Psi_1(\tau) = \frac{T_0 R^2 J_1(\mu_n)}{\mu_n} e^{-a\mu_n^2\tau/R^2}\left[1 - \sum_{m=1}^{\infty} A_m \cos \nu_m L e^{-a\nu_m^2\tau/L^2}\right],$$

$$A_m = \frac{2 \sin \nu_m}{\nu_m + \sin \nu_m \cos \nu_m} . \qquad (11)$$

We can find the original of the expression

$$\varphi_2(p) = \frac{h_1 \operatorname{sh} \gamma_n z + \gamma_n \operatorname{ch} \gamma_n z}{\gamma_n \omega(\gamma_n)} .$$

As before, putting $s = a\gamma_n^2$, we have

$$\Psi_2(\tau) = \frac{1}{2\pi i} e^{-a\mu_n^2\tau/R^2} \times$$

$$\times \int_{\sigma-i\infty}^{\sigma+i\infty} \frac{\sqrt{s/a} \operatorname{sh} \sqrt{s/a}\, z + s/(h_1 a) \operatorname{ch} \sqrt{s/a}}{\Omega(s)} e^{s\tau} ds . \qquad (12)$$

The denominator of the integrand in (12) has the same poles as the denominator of the integrand in (9). Thus we can write

$$\Psi_2(\tau) = \frac{a}{L} \sum_{m=1}^{\infty} A_m \nu_m [\sin \nu_m z/L +$$

$$+ \nu_m/(h_1 L) \cos \nu_m z/L] \exp\left[-a\left(\frac{\mu_n^2}{R^2} + \frac{\nu_m^2}{L^2}\right)\tau\right] . \qquad (13)$$

The last term in (5) is the product of two transformed variables. The original is determined through the application of the theorem on the multiplication of transforms and can be written in the form

$$\varphi_2(p) \int_0^R \xi J_0(\mu_n \xi/R) E_1(\xi, L, p)\, d\xi \doteqdot$$

$$\doteqdot \int_0^\tau \Psi_2(\tau - t)\, dt \int_0^R \xi J_0(\mu_n \xi/R) E(\xi, L, t)\, d\xi . \qquad (14)$$

We find the original of $\varphi_3(p)F(p)$, where

$$\varphi_3(p) = \left[\frac{R^2 J_1(\mu_n)}{\mu_n} - \frac{hRJ_0(\mu_n)}{\gamma_n^2}\right] \frac{h_1 \operatorname{ch} \gamma_n (L - z)}{\omega(\gamma_n)} .$$

We have

$$\Psi_3(\tau) = \frac{1}{2\pi i} e^{-a\mu_n^2\tau/R^2} \int_{\sigma-i\infty}^{\sigma+i\infty} [sR^2 J_1(\mu_n) -$$

$$- ha\mu_n J_0(\mu_n)] \operatorname{ch}[\sqrt{s/a}(L - z)] \frac{e^{s\tau}}{s\Omega(s)} ds . \qquad (15)$$

As before, the integral in (15) is equal to the sum of the residues at all the poles s_m ($m = 0, 1, 2, \ldots$), and so

$$\Psi_3(\tau) = -e^{-a\mu_n^2\tau/R^2}\Big\{ahRJ_0(\mu_n) -$$

$$- a \sum_{m=1}^{\infty} A_m [\nu_m^2 J_1(\mu_n)/(\mu_n L^2) +$$

$$+ hRJ_0(\mu_n)] \cos \nu_m L(1 - z/L)\, e^{-a\nu_m^2\tau/L^2}\Big\} . \qquad (16)$$

Then

$$\varphi_3(p) F(p) \doteqdot \int_0^\tau \Psi_3(\tau - t) f(t)\, dt .$$

The original of the second term in (5) is defined by

$$\frac{hRJ_0(\mu_n)}{\gamma_n^2} F(p) \doteqdot$$

$$\doteqdot ahRJ_0(\mu_n) \int_0^\tau \exp[-a\mu_n^2(\tau - t)/R^2] f(t)\, dt , \qquad (17)$$

Consider the particular case in which the base of the cylinder is thermally insulated. This corresponds to the case in which the condition $B_i = h_1 L$ tends to zero.

Then the roots of (10) are $\nu_m = \pi(m - 1)$ ($m = 1, 2, \ldots$), and so

$$\lim_{\nu_m \to (m-1)\pi} A_m =$$

$$= \lim_{\nu_m \to (m-1)\pi} \frac{2 \sin \nu_m}{\nu_m + \sin \nu_m \cos \nu_m} = \begin{cases} 1 & (m = 1) \\ 0 & (m > 1) . \end{cases} \qquad (18)$$

Taking note of this, it is easy to see that, as $B_i \to 0$,

$$\Psi_1(\tau) = \Psi_2(\tau) = \Psi_3(\tau) \to 0 . \qquad (19)$$

In this case, the temperature field can be written as

$$\theta(r, z, \tau) = 1 - k\tau/T_0 + \frac{kR^2}{4aT_0}\left[(1 - \rho^2) + \frac{2}{Rh}\right] -$$

$$- \frac{k}{T_0} \sum_{n=1}^{\infty} \frac{A_n}{x_n} J_0(\mu_n \rho) \exp(-x_n \tau) +$$

$$+ \eta \int_0^\tau \left\{1 + 2 \sum_{m=1}^{\infty} (-1)^m \exp[-y_m(\tau - t)] \cos \frac{\pi m z}{L}\right\} \times$$

$$\times \sum_{n=1}^{\infty} \frac{\mu_n A_n}{J_1(\mu_n)} \exp[-x_n(\tau - t)]\, dt \times$$

$$\times \int_0^1 \xi J_0(\mu_n \rho) J_0(\mu_n \xi) \varphi(\xi, L, t)\, d\xi . \qquad (20)$$

Here

$$x_n = a\mu_n^2/R^2, \quad y_m = a(\pi m/L)^2,$$

$$A_n = \frac{2J_1(\mu_n)}{\mu_n [J_0^2(\mu_n) + J_1^2(\mu_n)]},$$

$$\eta = \frac{aH(T_1^4 - T_0^4)}{R^2 L T_0}, \quad \theta(r, z, \tau) = \frac{T(r, z, \tau)}{T_0},$$

$$\varphi(r, L, \tau) = \frac{E(r, L, \tau)}{H[T_1^4 - T_0^4]} .$$

In (20), the resulting nondimensional radiation density $\varphi(r, L, \tau)$ is an unknown quantity and the temperature field may be formally represented in terms of it. As in the one-dimensional case [3], putting $z = L$ in (20), the problem can be reduced to a nonlinear integral equation for the resulting radiation density $\varphi(r, L, \tau)$ which can be put in the following form:

$$\varphi(\rho, \tau) = \delta - \gamma\Big\{1 - k\tau/T_0 + \frac{kR^2}{4aT_0}\left[(1 - \rho^2) + \frac{2}{Rh}\right] -$$

$$- \frac{k}{T_0} \sum_{n=1}^{\infty} \frac{A_n}{x_n} J_0(\mu_n \rho) \exp(-x_n \tau) +$$

$$+ \eta_0 \int_0^\tau \left[1 + 2 \sum_{n=1}^{\infty} \exp(-y_m(\tau - t))\right] \times$$

$$\times \sum_{n=1}^{\infty} \frac{\mu_n A_n}{J_1(\mu_n)} \exp(-x_n(\tau - t))\, dt \times$$

$$\times \int_0^1 \xi J_0(\mu_n \rho) J_0(\mu_n \xi) \varphi(\xi, t)\, d\xi\Big\}^4 . \qquad (21)$$

Here

$$\delta = T_1^4 / (T_1^4 - T_0^4), \quad \gamma = T_0^4 / (T_1^4 - T_0^4), \quad \eta_0 = \eta R^2 .$$

The integral equation (21) describes the heating of the surface $z = L$ of the cylinder. The numerical solution is obtained by Newton's iteration method [4] .

The numerical solution was arrived at as follows: the integral with respect to the space coordinate was divided into a finite sum by

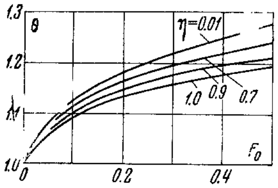

Fig. 1

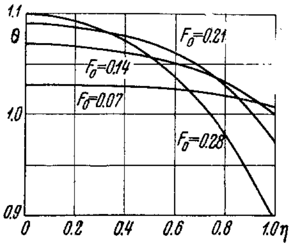

Fig. 2

Gauss's quadrature formula, and the integral with respect to time was divided into a finite sum by the rectangle formula. Thus, the temperature field $\theta(r, z, \tau)$ was defined after the values of the resulting flux density at the surface $z = L$ for any instant of time were found as a result of solving (21).

The relationship between the nondimensional temperature of the surface $z = L$ and the Fourier number for various radial sections with an initial cylinder temperature of $T_0 = 293°$ K is shown in Fig. 1.

The relationship between the nondimensional temperature of the thermally insulated base $z = 0$ of the cylinder and the nondimensional radius for various Fourier values and $B_i = 1$ is shown in Fig. 2 for an initial temperature of $T_0 = 293°$ K. The calculations were made for the following data: $R = 0.12$ m, $L = 0.015$ m, $a = 3.3 \cdot 10^{-3}$ m^2/hr, $H = 4.9 \cdot 10^{-9}$ deg^3/m.

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